

SOS model partition function and the elliptic weight functions

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Abstract. We generalize a recent observation [1] that the partition function of the 6-vertex model with domain-wall boundary conditions can be obtained by computing the projections of the product of the total currents in the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ in its current realization. A generalization is proved for the elliptic current algebra [2, 3]. The projections of the product of total currents are calculated explicitly and are represented as integral transforms of the product of the total currents. We prove that the kernel of this transform is proportional to the partition function of the SOS model with domain-wall boundary conditions.

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1. Introduction

The main aim of this paper is to apply the method of elliptic current projection to the computation of the universal elliptic weight functions. The projection of currents first appeared in the works of B. Enriquez and the second author [4], [5], as a method to construct a higher genus analog of the quantum groups in terms of Drinfeld currents [6]. The current (or “new”) realization supplies a quantum affine algebra with a second co-product, the “Drinfeld co-product”. The standard and Drinfeld co-products are related by a “twist” (see [4]). The quantum algebra is decomposed in two different ways a product of two Borel subalgebras. For each subalgebra, we can consider its intersection with these two Borel subalgebras and express it as their product. Thus we obtain for each subalgebra a pair of projection operators from it to each of these intersections. The above-mentioned twist is defined by a Hopf pairing of the subalgebras and the projection operators. See Section 4 where we recall an elliptic version of this construction.

S. Khoroshkin and the first author have applied this method to a factorization of the universal R -matrix [7] in quantum affine algebras, in order to obtain universal

weight functions [1, 8] for arbitrary quantum affine algebras. The weight functions play a fundamental role in the theory of deformed Knizhnik-Zamolodchikov and Knizhnik-Zamolodchikov-Bernard equations. In particular, in the case of $U_q(\widehat{\mathfrak{gl}}_n)$, acting by the projection of Drinfeld currents onto the highest weight vectors of irreducible finite-dimensional representations, one obtains exactly the (trigonometric) weight functions or off-shell Bethe vectors. In the canonical nested Bethe Ansatz, these objects are defined implicitly by recursive relations. Calculations of the projections are an effective way to determine the hierarchical relations of the nested Bethe Ansatz.

It was observed in [1] that the projections for the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ can be represented as integral transforms and that the kernels of these transforms are proportional to the partition function of the finite 6-vertex model with domain-wall boundary conditions (DWBC) [1]. We prove that the elliptic projections described in [2] make it possible to derive the partition function for elliptic models. We show that the calculation of the projections in the current elliptic algebra [2, 3] yields the partition function of the Solid-On-Solid (SOS) model with domain-wall boundary conditions.

The partition function for the finite 6-vertex model with domain wall boundary conditions was obtained by Izergin [9], who derived recursion relations for the partition function and solved them in determinant form. The kernels of the projections satisfy the same recursion relations and provide another formula for the partition function.

The problem of generalizing Izergin's determinant formula to the elliptic case has been extensively discussed in the last two decades. One can prove that the statistical sum of the SOS model with DWBC cannot be represented in the form of a single determinant. While this paper was in preparation, H. Rosengren [10] showed that this statistical sum for an $n \times n$ lattice can be written as a sum of 2^n determinants, thus generalizing Izergin's determinant formula. His approach relates to some dynamical generalization of the method of Alternating-Sign Matrices and follows the famous combinatorial proof of Kuperberg [16].

We expect that the projection method gives a universal form for the elliptic weight function [11] as it does for the quantum affine algebras [12]. When this universal weight function is represented as an integral transform of the product of the elliptic currents, we show that the kernel of this transform gives an expression of the partition function for the SOS model. On the one hand we generalize Izergin's recurrent relations and on the other hand we generalize to the elliptic case the method proposed in [1] for calculating the projections. We check that the kernel extracted from the universal weight function and multiplied by a certain factor satisfies the recursion relations that have been obtained, which uniquely define the partition function for the SOS model with DWBC. Our formula given by the projection method coincides with Rosengren's.

An interesting open problem which deserves more a extensive study is the relation of the projection method with the elliptic Sklyanin-Odesskii-Feigin algebras. It was observed in the pioneering paper [3] that half of the elliptic current generators satisfy the commutation relations of the W -elliptic algebras of Feigin. Another intriguing relation was observed in [17]: there exists a certain subalgebra in the " λ -generalization" of the

Sklyanin algebra such that its generators obey the Felder's R -matrix quadratic relations given in [18]. The latter paper gives also a description of the elliptic Bethe eigenvectors (the elliptic weight functions).

This is a strong indication that the projection method should be considered and interpreted in the framework of the (generalized) Sklyanin-Odesskii-Feigin algebras. We hope to discuss this problem elsewhere.

The main results of this paper were reported at the 7-th International Workshop on "Supersymmetry and Quantum Symmetry" in JINR, Dubna (Russia), July 30 - August 4, 2007.

The paper is organized as follows. In section 2 we briefly review the finite 6-vertex model with DWBC, and we present the formulae for the partition function: Izergin's determinant formula and the formula obtained by the projection method. Section 3 is devoted to the SOS model with DWBC. We briefly introduce the model and pose the problem of how to calculate the partition function of this model. We derive analytical properties of the partition function that allow us to reconstruct the partition function exactly. In section 4 we introduce the projections in terms of the currents for the elliptic algebra, following [2]. We generalize the method proposed in [1] to this case in order to obtain the integral representation of the projections of products of currents. Then, using a Hopf pairing, we extract the kernel and show that it satisfies all the necessary analytical properties of the partition function of the SOS model with DWBC. In Section 5, we investigate the trigonometric degeneration of the elliptic model and of the partition function with DWBC. We arrive at the 6-vertex model case in two steps. The model obtained after the first step is a trigonometric SOS model. Then we show that the degeneration of the expression derived in Section 4 coincides with the known expression for the 6-vertex model partition function with DWBC. An appendix contains the necessary information on the properties of elliptic polynomials.

2. Partition function of the finite 6-vertex model

Let us consider a statistical system on a square $n \times n$ lattice, where the columns and rows are numbered from 1 to n from right to left and from bottom to top, respectively. This is a 6-vertex model where the vertices on the lattice are associated with Boltzmann weights which depend on the configuration of the arrows around a given vertex. The six possible configurations are shown in Fig. 1, The weights are functions of two spectral parameters z , w and anisotropy parameter q :

$$\begin{aligned} a(z, w) &= qz - q^{-1}w, & b(z, w) &= z - w, \\ c(z, w) &= (q - q^{-1})z, & \bar{c}(z, w) &= (q - q^{-1})w. \end{aligned} \tag{1}$$

Let us associate the sign '+' to the arrows directed upward and to the left, while the sign '-' is associated to the arrows directed downward and to the right as shown

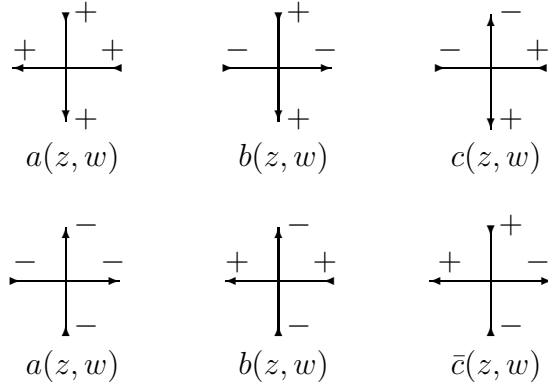


Figure 1. Graphical presentation of the Boltzmann weights.

in Fig. 1. The Boltzmann weights (1) are gathered in the matrix

$$R(z, w) = \begin{pmatrix} a(z, w) & 0 & 0 & 0 \\ 0 & b(z, w) & \bar{c}(z, w) & 0 \\ 0 & c(z, w) & b(z, w) & 0 \\ 0 & 0 & 0 & a(z, w) \end{pmatrix} \quad (2)$$

acting in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the basis $e_\alpha \otimes e_\beta$, $\alpha, \beta = \pm$. The entry $R(z, w)_{\gamma\delta}^{\alpha\beta}$, $\alpha, \beta, \gamma, \delta = \pm$ coincides with the Boltzmann weight corresponding to Fig. 2:

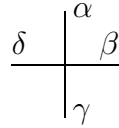


Figure 2. The Boltzmann weight $R(z, w)_{\gamma\delta}^{\alpha\beta}$.

Different repartitions of the arrows on the edges form different configurations $\{C\}$. A Boltzmann weight of the lattice is the product of the Boltzmann weights at each vertex. We define the partition function of the model as the sum of the Boltzmann weights of the lattice over all possible configurations, subject to some boundary conditions:

$$Z(\{z\}, \{w\}) = \sum_{\{C\}} \prod_{i,j=1}^n R(z_i, w_j)_{\gamma_{ij}\delta_{ij}}^{\alpha_{ij}\beta_{ij}}. \quad (3)$$

Here α_{ij} , β_{ij} , γ_{ij} , δ_{ij} are the signs corresponding to the arrows around the (i, j) -th vertex. We consider an inhomogeneous model where the Boltzmann weights depend on the column by the variable z_i and on the row by the variable w_j (see Fig. 3).

We choose the so-called domain-wall boundary conditions (DWBC) that fix the boundary arrows (signs) as shown in Fig. 3. In other words, the arrows enter on the left and right boundaries and leave on the lower and upper ones.

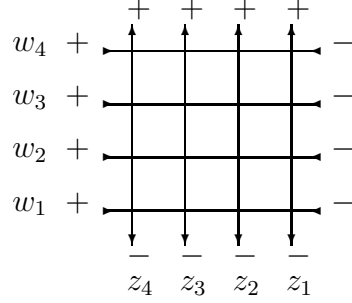


Figure 3. Inhomogeneous lattice with domain wall boundary conditions.

In [9], A. G. Izergin found a determinant representation for the partition function of the lattice with DWBC,

$$Z(\{z\}, \{w\}) = (q - q^{-1})^n \prod_{m=1}^n w_m \times \prod_{i,j=1}^n (z_i - w_j)(qz_i - q^{-1}w_j) \times \frac{\det \left\| \frac{1}{(z_i - w_j)(qz_i - q^{-1}w_j)} \right\|_{i,j=1,\dots,n}}{\prod_{n \geq i > j \geq 1} (z_i - z_j)(w_j - w_i)}. \quad (4)$$

Izergin's idea was to prove a symmetry of the polynomial (3), and then use it to find recursion relations for the quantity $Z(\{z\}, \{w\})$ and to observe that these recursion relations allow the reconstruction of $Z(\{z\}, \{w\})$ in a unique way and that the same recursion relations are valid for the determinant formula (4).

On the other hand it was observed that the kernel of the projection of n currents is a polynomial of the same degree, and satisfies the same recursion relations [1]. It means that this kernel coincides with the partition function for the $n \times n$ lattice. Moreover, the theory of projections gives another expression for the partition function:

$$Z(\{z\}, \{w\}) = (q - q^{-1})^n \prod_{m=1}^n w_m \prod_{n \geq i > j \geq 1} \frac{q^{-1}w_i - qw_j}{w_i - w_j} \times \sum_{\sigma \in S_n} \prod_{\substack{1 \leq i < j \leq n \\ \sigma(i) > \sigma(j)}} \frac{qw_{\sigma(i)} - q^{-1}w_{\sigma(j)}}{q^{-1}w_{\sigma(i)} - qw_{\sigma(j)}} \prod_{n \geq i > k \geq 1} (qz_i - q^{-1}w_{\sigma(k)}) \prod_{1 \leq i < k \leq n} (z_i - w_{\sigma(k)}), \quad (5)$$

where S_n is the group of permutations. Here the factor $\frac{qw_{\sigma(i)} - q^{-1}w_{\sigma(j)}}{q^{-1}w_{\sigma(i)} - qw_{\sigma(j)}}$ appears in the product if both conditions $i < j$ and $\sigma(i) > \sigma(j)$ are satisfied simultaneously.

3. Partition function for the SOS model

3.1. Description of the SOS model

The SOS model is a face model. We introduce it in terms of heights as usual, but then we represent it in the R -matrix formalism as in [13]. This language is more convenient to

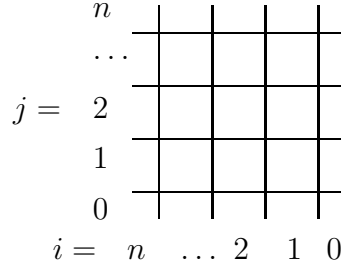


Figure 4. The numbering of faces.

generalize the results reviewed in Section 2 and to prove the symmetry of the partition function.

Consider a square $n \times n$ lattice with vertices enumerated by the index $i = 1, \dots, n$ as in the previous case. It has $(n+1) \times (n+1)$ faces enumerated by pairs (i, j) , $i, j = 0, \dots, n$ (see Fig. 4). To each face we assign a complex number called its height in such a way that the differences of the heights corresponding to the neighboring faces are ± 1 . Let us denote by d_{ij} the height corresponding to the face (i, j) placed to the upper left of the vertex (i, j) . Then the last condition can be written in the form $|d_{ij} - d_{i-1,j}| = 1$, for $i = 1, \dots, n$, $j = 0, \dots, n$, and $|d_{ij} - d_{i,j-1}| = 1$ for $i = 0, \dots, n$, $j = 1, \dots, n$. Each distribution of heights d_{ij} ($i, j = 0, \dots, n$) subject to these conditions and to boundary conditions defines a configuration of the model. It means that the partition function of this model can be represented in the form

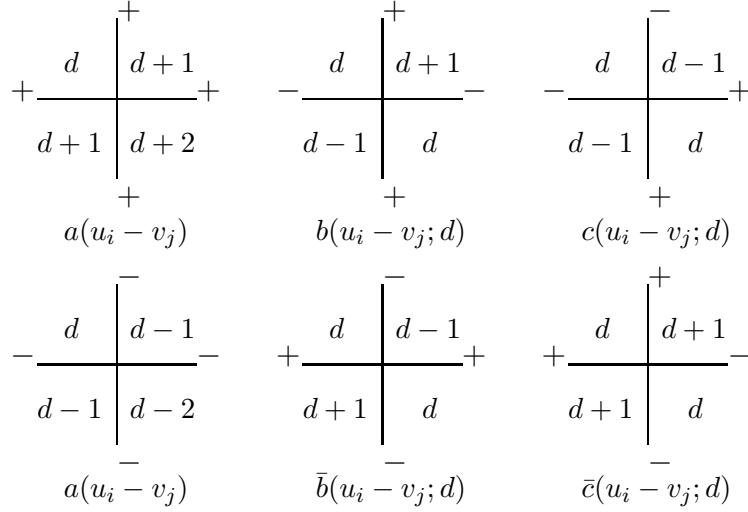
$$Z = \sum_C \prod_{i,j=1}^n W_{ij}(d_{i,j-1}, d_{i-1,j-1}, d_{i-1,j}, d_{ij}), \quad (6)$$

where $W_{ij}(d_{i,j-1}, d_{i-1,j-1}, d_{i-1,j}, d_{ij})$ is the Boltzmann weight of the (i, j) -th vertex depending on the configuration by means of the heights of the neighboring faces as follows [14]

$$\begin{aligned} W_{ij}(d+1, d+2, d+1, d) &= a(u_i - v_j) = \theta(u_i - v_j + \hbar), \\ W_{ij}(d-1, d-2, d-1, d) &= a(u_i - v_j) = \theta(u_i - v_j + \hbar), \\ W_{ij}(d-1, d, d+1, d) &= b(u_i - v_j; \hbar d) = \frac{\theta(u_i - v_j)\theta(\hbar d + \hbar)}{\theta(\hbar d)}, \\ W_{ij}(d+1, d, d-1, d) &= \bar{b}(u_i - v_j; \hbar d) = \frac{\theta(u_i - v_j)\theta(\hbar d - \hbar)}{\theta(\hbar d)}, \\ W_{ij}(d-1, d, d-1, d) &= c(u_i - v_j; \hbar d) = \frac{\theta(u_i - v_j + \hbar d)\theta(\hbar)}{\theta(\hbar d)}, \\ W_{ij}(d+1, d, d+1, d) &= \bar{c}(u_i - v_j; \hbar d) = \frac{\theta(u_i - v_j - \hbar d)\theta(\hbar)}{\theta(-\hbar d)}. \end{aligned} \quad (7)$$

As in the 6-vertex case the variables u_i, v_j are attached to the i -th vertical and j -th horizontal lines respectively, \hbar is a nonzero anisotropy parameter \ddagger . The weights are

\ddagger In the elliptic case, we use additive variables u_i, v_j and an additive anisotropy parameter \hbar instead of the multiplicative variables $z_i = e^{2\pi i u_i}$, $w_i = e^{2\pi i v_i}$ and the multiplicative parameter $q = e^{\pi i \hbar}$.

**Figure 5.** The Boltzmann weights for the SOS model.

expressed by means of the ordinary odd theta-function defined by the conditions

$$\theta(u+1) = -\theta(u), \quad \theta(u+\tau) = -e^{-2\pi i u - \pi i \tau} \theta(u), \quad \theta'(0) = 1. \quad (8)$$

Let us introduce the notations

$$\alpha_{ij} = d_{i-1,j} - d_{ij}, \quad \beta_{ij} = d_{i-1,j-1} - d_{i-1,j}, \quad \gamma_{ij} = d_{i-1,j-1} - d_{i,j-1}, \quad \delta_{ij} = d_{i,j-1} - d_{ij}. \quad (9)$$

The differences (9) take the values ± 1 and we attach them to the corresponding edges as in Fig. 2: $\gamma_{i,j+1} = \alpha_{ij}$ is the sign attached to the vertical edge connecting the (i, j) -th vertex to the $(i, j+1)$ -st one, $\beta_{i,j+1} = \delta_{ij}$ is the sign attached to the horizontal edge connecting the (i, j) -th vertex to the $(i+1, j)$ -th one. The configuration can be considered as a distribution of these signs on the internal edges subject to the conditions $\alpha_{ij} + \beta_{ij} = \gamma_{ij} + \delta_{ij}$, $i, j = 1, \dots, n$. In terms of signs on the external edges the DWBC are the same as shown in Fig. 3. Additionally, we have to fix one of the boundary heights, for example, d_{nn} .

The Boltzmann weights (7) can be represented as the entries of a dynamical elliptic R -matrix [13]:

$$W_{ij}(d_{i,j-1}, d_{i-1,j-1}, d_{i-1,j}, d_{ij}) = R(u_i - v_j; \hbar d_{ij})_{\gamma_{ij}\delta_{ij}}^{\alpha_{ij}\beta_{ij}},$$

$$R(u; \lambda) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u; \lambda) & \bar{c}(u; \lambda) & 0 \\ 0 & c(u; \lambda) & \bar{b}(u; \lambda) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}. \quad (10)$$

Let $\mathbb{T}_{\gamma_{i1}}^{\alpha_{in}}(u_i, \{v\}, \lambda_i)$ be the *column transfer matrix*. It is a matrix-valued function of u_i , all spectral parameters v_j , $j = 1, \dots, n$ and the parameters λ_i related to the heights:

$$\begin{aligned} \mathbb{T}_{\gamma_{i1}}^{\alpha_{in}}(u_i, \{v\}, \lambda_i)_{\delta_{in} \dots \delta_{i1}}^{\beta_{in} \dots \beta_{i1}} &= \\ &= \left(R^{(n+1,n)}(u_i - v_n; \lambda_{in}) R^{(n+1,n-1)}(u_i - v_{n-1}; \lambda_{i,n-1}) \dots R^{(n+1,1)}(u_i - v_1; \lambda_{i1}) \right)_{\gamma_{i1}; \delta_{in} \dots \delta_{i1}}^{\alpha_{in}; \beta_{in} \dots \beta_{i1}} \end{aligned} \quad (11)$$

$$= \left(R^{(n+1,n)}(u_i - v_n; \Lambda_{in}) R^{(n+1,n-1)}(u_i - v_{n-1}; \Lambda_{i,n-1}) \cdots R^{(n+1,1)}(u_i - v_1; \Lambda_{i1}) \right)_{\gamma_{i1}; \delta_{in} \dots \delta_{i1}}^{\alpha_{in}; \beta_{in} \dots \beta_{i1}},$$

where $\lambda_{ij} = \hbar d_{ij} = \lambda_i + \hbar \sum_{l=j+1}^n \delta_{il}$, $\lambda_i = \hbar d_{in} = \lambda + \hbar \sum_{l=i+1}^n \alpha_{ln}$, $\Lambda_{ij} = \lambda_i + \hbar \sum_{l=j+1}^n H^{(l)}$.

The matrix $H^{(l)}$ acts in the l -th two-dimensional space $V_l \cong \mathbb{C}^2$ as $\text{diag}(1, -1)$ and the R -matrix $R^{(a,b)}$ acts nontrivially in the tensor product $V_a \otimes V_b$. The superscript $n+1$ in the R -matrices is regarded as belonging to an auxiliary space $V_{n+1} \cong \mathbb{C}^2$. The partition function (6) corresponding to DWBC ($\alpha_{in} = +1$, $\beta_{1i} = -1$, $\gamma_{i1} = -1$, $\delta_{ni} = +1$, $i = 1, \dots, n$) can be represented by means of the column transfer matrices:

$$Z_{-+}^{\pm}(\{u\}, \{v\}; \lambda) = \left(\mathbb{T}_{-}^{\pm}(u_1, \{v\}, \lambda_1) \cdots \mathbb{T}_{-}^{\pm}(u_n, \{v\}, \lambda_n) \right)_{+ \dots +}^{- \dots -}, \quad (12)$$

where $\lambda_i = \lambda + \hbar(n - i)$. Similarly one can define the row transfer matrix.

3.2. Analytical properties of the partition function

Here we describe the analytical properties of the SOS model partition function which are analogous to those used by A.G. Izergin in order to recover the partition function of the 6-vertex model. These properties uniquely define this partition function.

Proposition 1 *The partition function with DWBC $Z_{-+}^{\pm}(\{u\}, \{v\}; \lambda)$ is a symmetric function in both sets of variables u_i and v_j .*

The proof is based on the dynamical Yang-Baxter equation (DYBE) for the R -matrix [13]

$$\begin{aligned} R^{(12)}(t_1 - t_2; \lambda) R^{(13)}(t_1 - t_3; \lambda + \hbar H^{(2)}) R^{(23)}(t_2 - t_3; \lambda) = \\ = R^{(23)}(t_2 - t_3; \lambda + \hbar H^{(1)}) R^{(13)}(t_1 - t_3; \lambda) R^{(12)}(t_1 - t_2; \lambda + \hbar H^{(3)}). \end{aligned}$$

In order to prove the symmetry of the partition function $Z_{-+}^{\pm}(\{u\}, \{v\}; \lambda)$ under the permutation $v_j \leftrightarrow v_{j-1}$, we rewrite the DYBE in the form

$$\begin{aligned} R^{(n+1,j)}(u_i - v_j; \Lambda_{ij}) R^{(n+1,j-1)}(u_i - v_{j-1}; \Lambda_{ij} + \hbar H^{(j)}) \\ \times R^{(j,j-1)}(v_j - v_{j-1}; \Lambda_{ij}) = R^{(j,j-1)}(v_j - v_{j-1}; \Lambda_{ij} + \hbar H^{(n+1)}) \\ \times R^{(n+1,j-1)}(u_i - v_{j-1}; \Lambda_{ij}) R^{(n+1,j)}(u_i - v_j; \Lambda_{ij} + \hbar H^{(j-1)}). \end{aligned} \quad (13)$$

Multiplying the i -th column matrix (11) by $R^{(j,j-1)}(v_j - v_{j-1}; \Lambda_{ij})$ to the right and moving it to the left using (13), the relation $[H_1 + H_2, R(u, \lambda)] = 0$ and the equality $\Lambda_{ij} + \hbar \alpha_{in} = \Lambda_{i-1,j}$, we obtain

$$\begin{aligned} \mathbb{T}_{\gamma_{i1}}^{\alpha_{in}}(u_i, \{v\}, \lambda_i) R^{(j,j-1)}(v_j - v_{j-1}; \Lambda_{ij}) = R^{(j,j-1)}(v_j - v_{j-1}; \Lambda_{i-1,j}) \\ \times \mathcal{P}^{(j,j-1)} \mathbb{T}_{\gamma_{i1}}^{\alpha_{in}}(u_i, \{v_j \leftrightarrow v_{j-1}\}, \lambda_i) \cdots \mathbb{T}_{\gamma_{n1}}^{\alpha_{nn}}(u_n, \{v_j \leftrightarrow v_{j-1}\}, \lambda_n) \mathcal{P}^{(j,j-1)}, \end{aligned} \quad (14)$$

where $\mathcal{P} \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is a permutation matrix: $\mathcal{P}(e_1 \otimes e_2) = e_2 \otimes e_1$ for all $e_1, e_2 \in \mathbb{C}^2$ and the notation $\{v_j \leftrightarrow v_{j-1}\}$ means that the set of parameters $\{v\}$ with v_{j-1} and v_j are interchanged. Multiplying the product of the column matrix by $R^{(j,j-1)}(v_j - v_{j-1}; \Lambda_{nj})$ to the right and moving it to the left using (14) one yields

$$\begin{aligned} \mathbb{T}_{\gamma_{11}}^{\alpha_{1n}}(u_1, \{v\}, \lambda_1) \cdots \mathbb{T}_{\gamma_{n1}}^{\alpha_{nn}}(u_n, \{v\}, \lambda_n) R^{(j,j-1)}(v_j - v_{j-1}; \Lambda_{nj}) = R^{(j,j-1)}(v_j - v_{j-1}; \Lambda_{0,j}) \\ \times \mathcal{P}^{(j,j-1)} \mathbb{T}_{\gamma_{11}}^{\alpha_{1n}}(u_1, \{v_j \leftrightarrow v_{j-1}\}, \lambda_1) \cdots \mathbb{T}_{\gamma_{n1}}^{\alpha_{nn}}(u_n, \{v_j \leftrightarrow v_{j-1}\}, \lambda_n) \mathcal{P}^{(j,j-1)}, \end{aligned} \quad (15)$$

where $\Lambda_{0,j} = \lambda + \hbar \sum_{i=1}^n \alpha_{in} + \hbar \sum_{l=j+1}^n H^{(l)}$, $\Lambda_{nj} = \lambda_n = \lambda$. Finally, comparing the matrix element $(\cdot)_{+, \dots, +}^{-, \dots, -}$ of both sides of (15), taking into account Formula (12) and the identities $R(u, \lambda)_{\gamma\delta}^{-} = a(u)\delta_{\gamma}^{-}\delta_{\delta}^{-}$, $R(u, \lambda)_{++}^{\alpha\beta} = a(u)\delta_{+}^{\alpha}\delta_{+}^{\beta}$, $\mathcal{P}_{\gamma\delta}^{-} = \delta_{\gamma}^{-}\delta_{\delta}^{-}$, $\mathcal{P}_{++}^{\alpha\beta} = \delta_{+}^{\alpha}\delta_{+}^{\beta}$, (where δ_{γ}^{α} is the Kronecker symbol) and substituting $\alpha_{in} = +1$, $\gamma_{i1} = -1$, one derives

$$Z_{-+}^{+-}(\{u\}, \{v\}; \lambda) = Z_{-+}^{+-}(\{u\}, \{v_j \leftrightarrow v_{j-1}\}; \lambda). \quad (16)$$

Similarly, using the row transfer matrix one can obtain the following equality from the DYBE:

$$Z_{-+}^{+-}(\{u\}, \{v\}; \lambda) = Z_{-+}^{+-}(\{u_j \leftrightarrow u_{j-1}\}, \{v\}; \lambda). \quad (17)$$

The partition function with DWBC satisfies relations (16), (17) for each $j = 1, \dots, n$, which is sufficient to establish the symmetry under an arbitrary permutation. \square

Proposition 2 *The partition function with DWBC (12) is an elliptic polynomial § of degree n with character χ in each variable u_i , where*

$$\chi(1) = (-1)^n, \quad \chi(\tau) = (-1)^n \exp\left(2\pi i\left(\lambda + \sum_{j=1}^n v_j\right)\right). \quad (18)$$

Due to the symmetry with respect to the variables $\{u\}$ it is sufficient to prove the proposition for the variable u_n . To represent explicitly the dependence of $Z_{-+}^{+-}(\{u\}, \{v\}; \lambda)$ on u_n , we consider all the possibilities for the states of the edges attached to the vertices located in the n -th column. First, consider the (n, n) -th vertex. Due to the boundary conditions $\alpha_{nn} = \delta_{nn} = +1$ and to the condition $\alpha_{nn} + \beta_{nn} = \gamma_{nn} + \delta_{nn}$ we have two possibilities: either $\beta_{nn} = \gamma_{nn} = -1$ or $\beta_{nn} = \gamma_{nn} = +1$. In the first case, one has a unique possibility for the rest of the n -th column: $\gamma_{nj} = -1$, $\beta_{nj} = +1$, $j = 1, \dots, n-1$; in the second case, there are two possibilities for the $(n, n-1)$ -st vertex: either $\beta_{n,n-1} = \gamma_{n,n-1} = -1$ or $\beta_{n,n-1} = \gamma_{n,n-1} = +1$, etc. Finally the partition function is represented in the form

$$\begin{aligned} Z_{-+}^{+-}(\{u\}, \{v\}; \lambda) &= \sum_{k=1}^n \prod_{j=k+1}^n a(u_n - v_j) \bar{c}(u_n - v_k; \lambda + (n-k)\hbar) \\ &\quad \times \prod_{j=1}^{k-1} \bar{b}(u_n - v_j; \lambda + (n-j)\hbar) g_k(u_{n-1}, \dots, u_1, \{v\}; \lambda), \end{aligned}$$

where $g_k(u_{n-1}, \dots, u_1, \{v\}; \lambda)$ are functions which do not depend on u_n . Each term of this sum is an elliptic polynomial of degree n with the same character (18) in the variable u_n . \square

Remark 1 *Similarly, one can prove that the function $Z_{-+}^{+-}(\{u\}, \{v\}; \lambda)$ is an elliptic polynomial of degree n with character $\tilde{\chi}$ in each variable v_i , where $\tilde{\chi}(1) = (-1)^n$, $\tilde{\chi}(\tau) = (-1)^n e^{2\pi i(-\lambda + \sum_{i=1}^n u_i)}$.*

§ The definition of elliptic polynomials and their properties are given in Appendix A.

Proposition 3 *The n -th partition function with DWBC (12) with the condition $u_n = v_n - \hbar$ can be expressed through the $(n-1)$ -st partition function:*

$$\begin{aligned} Z_{-+}^{+-}(u_n = v_n - \hbar, u_{n-1}, \dots, u_1; v_n, v_{n-1}, \dots, v_1; \lambda) = \\ = \frac{\theta(\lambda + n\hbar)\theta(\hbar)}{\theta(\lambda + (n-1)\hbar)} \prod_{m=1}^{n-1} \left(\theta(v_n - v_m - \hbar)\theta(u_m - v_n) \right) Z_{-+}^{+-}(u_{n-1}, \dots, u_1; v_{n-1}, \dots, v_1; \lambda). \end{aligned} \quad (19)$$

Considering the n -th column and the n -th row and taking into account that $a(u_n - v_n)|_{u_n=v_n-\hbar} = a(-\hbar) = 0$ we conclude that the unique possibility for a non-trivial contribution is: $\beta_{nn} = \gamma_{nn} = -1$, $\gamma_{nj} = -1$, $\beta_{nj} = +1$, $j = 1, \dots, n-1$, $\beta_{in} = -1$, $\gamma_{in} = +1$, $i = 1, \dots, n-1$. The last formulae impose the same DWBC for the $(n-1) \times (n-1)$ sublattice: $\delta_{n-1,j} = \beta_{nj} = +1$, $j = 1, \dots, n-1$, $\alpha_{i,n-1} = \gamma_{in} = +1$, $i = 1, \dots, n-1$, $d_{n-1,n-1} = d_{nn}$. Thus the substitution $u_n = v_n - \hbar$ to the partition function for the whole lattice yields

$$\begin{aligned} Z_{-+}^{+-}(u_n = v_n - \hbar, u_{n-1}, \dots, u_1; v_n, v_{n-1}, \dots, v_1; \lambda) \\ = \bar{c}(-\hbar; \lambda) \prod_{j=1}^{n-1} \bar{b}(v_n - v_j - \hbar; \lambda + (n-j)\hbar) \prod_{i=1}^{n-1} b(u_i - v_n; \lambda + (n-i)\hbar) \\ \times Z_{-+}^{+-}(u_{n-1}, \dots, u_1; v_{n-1}, \dots, v_1; \lambda). \end{aligned} \quad (20)$$

Using the explicit expressions (7) for the Boltzmann weights, one can rewrite the last formula in the form (19). \square

Remark 2 *From Formula (20) we see that the following transformation of the R -matrix*

$$b(u, v; \lambda) \rightarrow \rho b(u, v; \lambda), \quad \bar{b}(u, v; \lambda) \rightarrow \rho^{-1} \bar{b}(u, v; \lambda) \quad (21)$$

does not change the recursion relation (19), where ρ is a non-zero constant which does not depend on u , v and λ .

Lemma 1 *If the set of functions $\{Z^{(n)}(u_n, \dots, u_1; v_n, \dots, v_1; \lambda)\}_{n \geq 1}$ satisfies the conditions of Propositions 1, 2, 3 and the initial condition*

$$Z^{(1)}(u_1; v_1; \lambda) = \bar{c}(u_1 - v_1) = \frac{\theta(u_1 - v_1 - \lambda)\theta(\hbar)}{\theta(-\lambda)} \quad (22)$$

then

$$Z_{-+}^{+-}(u_n, \dots, u_1; v_n, \dots, v_1; \lambda) = Z^{(n)}(u_n, \dots, u_1; v_n, \dots, v_1; \lambda). \quad (23)$$

Due to (22), this lemma can be proved by induction on n . Let the equality (23) be valid for $n-1$. Consider the functions $Z_{-+}^{+-}(u_n, \dots, u_1; v_n, \dots, v_1; \lambda)$ and $Z^{(n)}(u_n, \dots, u_1; v_n, \dots, v_1; \lambda)$ as functions of u_n . Both are elliptic polynomials of degree n with character (18). They have the same value at the point $u_n = v_n - \hbar$, and due to the symmetry of these functions with respect to the parameters $\{v_j\}_{j=1}^n$ they coincide at all points $u_n = v_j - \hbar$, $j = 1, \dots, n$. It follows from Lemma 2 (see Appendix A) that these functions are identical. \square

Remark 3 *As we can see from the proof of Lemma 1, it is sufficient to establish the symmetry only with respect to the variables v_j .*

Remark 4 The transformation (21) of the R -matrix does not change the partition function with DWBC.

4. Elliptic projections of currents

Let $\mathcal{K}_0 = \mathbb{C}[u^{-1}][[u]]$ be the completed space of complex-valued meromorphic functions defined in the neighborhood of the origin which have only simple poles at this point. Let $\{\epsilon^i\}$ and $\{\epsilon_i\}$ be dual bases in \mathcal{K}_0 such that $\oint \frac{du}{2\pi i} \epsilon^i(u) \epsilon_j(u) = \delta_j^i$.

4.1. Current description of the elliptic algebra

Let \mathcal{A} be a Hopf algebra generated by elements $\hat{h}[s]$, $\hat{e}[s]$, $\hat{f}[s]$, $s \in \mathcal{K}_0$, subject to the linear relations

$$\hat{x}[\alpha_1 s_1 + \alpha_2 s_2] = \alpha_1 \hat{x}[s_1] + \alpha_2 \hat{x}[s_2], \quad \alpha_1, \alpha_2 \in \mathbb{C}, \quad s_1, s_2 \in \mathcal{K}_0,$$

where $x \in \{h, e, f\}$. The commutation relations will be written in terms of the currents,

$$\begin{aligned} h^+(u) &= \sum_{i \geq 0} \hat{h}[\epsilon^{i;0}] \epsilon_{i;0}(u), & h^-(u) &= - \sum_{i \geq 0} \hat{h}[\epsilon_{i;0}] \epsilon^{i;0}(u), \\ f(u) &= \sum_i \hat{f}[\epsilon^i] \epsilon_i(u), & e(u) &= \sum_i \hat{e}[\epsilon^i] \epsilon_i(u). \end{aligned} \quad (24)$$

The currents $e(u)$ and $f(u)$ are called the *total currents*. They are defined in terms of dual bases of \mathcal{K}_0 and their definition does not depend on the choice of these dual bases (see [3, 15]). The currents $h^+(u)$ and $h^-(u)$ are called the *Cartan currents* and they are defined in terms of the special basis

$$\epsilon^{k;0}(u) = \frac{1}{k!} \left(\frac{\theta'(u)}{\theta(u)} \right)^{(k)}, \quad k \geq 0; \quad \epsilon_{k;0}(u) = (-u)^k, \quad k \geq 0.$$

The commutation relations are [2]:

$$\begin{aligned} [K^\pm(u), K^\pm(v)] &= 0, & [K^+(u), K^-(v)] &= 0, \\ K^\pm(u) e(v) K^\pm(u)^{-1} &= \frac{\theta(u-v+\hbar)}{\theta(u-v-\hbar)} e(v), \\ K^\pm(u) f(v) K^\pm(u)^{-1} &= \frac{\theta(u-v-\hbar)}{\theta(u-v+\hbar)} f(v), \\ \theta(u-v-\hbar) e(u) e(v) &= \theta(u-v+\hbar) e(v) e(u), \end{aligned} \quad (25)$$

$$\theta(u-v+\hbar) f(u) f(v) = \theta(u-v-\hbar) f(v) f(u), \quad (26)$$

$$[e(u), f(v)] = \hbar^{-1} \delta(u, v) (K^+(u) - K^-(v)),$$

where $K^+(u) = \exp\left(\frac{e^{\hbar\partial u} - e^{-\hbar\partial u}}{2\partial u} h^+(u)\right)$, $K^-(u) = \exp(\hbar h^-(u))$ and $\delta(u, v) = \sum_{n \in \mathbb{Z}} \frac{u^n}{v^{n+1}}$ is a delta-function || for \mathcal{K}_0 . The algebra \mathcal{A} is a non-central version of the algebra $A(\tau)$ introduced in [3]. This algebra is equipped with the co-product and co-unit:

$$\begin{aligned}\Delta K^\pm(u) &= K^\pm(u) \otimes K^\pm(u), \\ \Delta e(u) &= e(u) \otimes 1 + K^-(u) \otimes e(u), \\ \Delta f(u) &= f(u) \otimes K^+(u) + 1 \otimes f(u), \\ \varepsilon(K^\pm(u)) &= 1, \quad \varepsilon(e(u)) = 0, \quad \varepsilon(f(u)) = 0.\end{aligned}$$

Let \mathcal{A}_F and \mathcal{A}_E be the subalgebras of \mathcal{A} generated by the generators $\hat{h}[\epsilon_{i;0}]$, $\hat{f}[s]$, and $\hat{h}[\epsilon^{i;0}]$, $\hat{e}[s]$, respectively, $s \in \mathcal{K}_0$. The subalgebra \mathcal{A}_F is described by the currents $K^+(u)$, $f(u)$, and the subalgebra \mathcal{A}_E by $K^-(u)$, $e(u)$. We introduce the notation H^+ for the subalgebra of \mathcal{A} generated by $\hat{h}[\epsilon_{i;0}]$. As stated in [2], the bialgebras $(\mathcal{A}_F, \Delta^{\text{op}})$ and (\mathcal{A}_E, Δ) are dual with respect to the Hopf pairing $\langle \cdot, \cdot \rangle: \mathcal{A}_F \times \mathcal{A}_E \rightarrow \mathbb{C}$ defined in terms of currents as follows:

$$\langle f(u), e(v) \rangle = \hbar^{-1} \delta(u, v), \quad \langle K^+(u), K^-(v) \rangle = \frac{\theta(u - v - \hbar)}{\theta(u - v + \hbar)}. \quad (27)$$

These formulae uniquely define a Hopf pairing on $\mathcal{A}_F \times \mathcal{A}_E$. In particular, one can derive the following formula

$$\begin{aligned}\langle f(t_n) \cdots f(t_1), e(v_n) \cdots e(v_1) \rangle &= \\ &= \hbar^{-n} \sum_{\sigma \in S_n} \prod_{\substack{l < l' \\ \sigma(l) > \sigma(l')}} \frac{\theta(v_{\sigma(l)} - v_{\sigma(l')} + \hbar)}{\theta(v_{\sigma(l)} - v_{\sigma(l')} - \hbar)} \prod_{m=1}^n \delta(t_m, v_{\sigma(m)}).\end{aligned} \quad (28)$$

4.2. Projections of currents

We define the projections as linear maps acting on the subalgebra \mathcal{A}_F . Dual projections, which we do not consider here, act in the subalgebra \mathcal{A}_E . We define the projections in terms of the *half-currents* $f_\lambda^+(u)$ and $f_\lambda^-(u)$, defined below. These are usually defined as parts of the sum (24) (with the corresponding sign) such that $f(u) = f_\lambda^+(u) - f_\lambda^-(u)$. Here λ is the parameter for the decomposition of the total current into the difference of half-currents. Elliptic half-currents are investigated in details on the classical level in [15]. We will introduce the half-currents by their representations by means of integral transforms of the total current $f(u)$:

$$f_\lambda^+(u) = \oint_{|v| < |u|} \frac{dv}{2\pi i} \frac{\theta(u - v - \lambda)}{\theta(u - v)\theta(-\lambda)} f(v), \quad f_\lambda^-(u) = \oint_{|v| > |u|} \frac{dv}{2\pi i} \frac{\theta(u - v - \lambda)}{\theta(u - v)\theta(-\lambda)} f(v), \quad (29)$$

where $\lambda \notin \Gamma = \mathbb{Z} + \mathbb{Z}\tau$. The half-current $f_\lambda^+(u)$ is called positive and $f_\lambda^-(u)$ is called negative.

|| One can find more details about distributions acting on \mathcal{K}_0 and their significance in the theory of current algebras in our previous paper [15].

The corresponding positive and negative projections are also parameterized by λ and they are defined on the half-currents as follows:

$$P_\lambda^+(f_\lambda^+(u)) = f_\lambda^+(u), \quad P_\lambda^-(f_\lambda^+(u)) = 0, \quad (30)$$

$$P_\lambda^+(f_\lambda^-(u)) = 0, \quad P_\lambda^-(f_\lambda^-(u)) = f_\lambda^-(u). \quad (31)$$

Let us first define the projections in the subalgebra \mathcal{A}_f generated by the currents $f(u)$. As a linear space this subalgebra is spanned by the products $f(u_n)f(u_{n-1})\cdots f(u_1)$, $n = 0, 1, 2, \dots$. It means that any element of \mathcal{A}_f can be represented as a sum (maybe infinite) of integrals \P

$$\oint \frac{du_n \cdots du_1}{(2\pi i)^n} f(u_n) \cdots f(u_1) s_n(u_n) \cdots s_1(u_1), \quad s_n, \dots, s_1 \in \mathcal{K}_0.$$

It follows from the PBW theorem proved in [2] that any element of \mathcal{A}_f can also be represented as the sum of the integrals

$$\oint \frac{du_n \cdots du_1}{(2\pi i)^n} f_{\lambda+2(n-1)\hbar}^-(u_n) \cdots f_{\lambda+2m\hbar}^-(u_{m+1}) f_{\lambda+2(m-1)\hbar}^+(u_m) \cdots f_\lambda^+(u_1) s_n(u_n) \cdots s_1(u_1),$$

$s_n \cdots s_1 \in \mathcal{K}_0$, $0 \leq m \leq n$. Therefore, it is sufficient to define the projections on these products of half-currents:

$$P_\lambda^+(x^- x^+) = \varepsilon(x^-) x^+, \quad P_\lambda^-(y^- y^+) = y^- \varepsilon(y^+), \quad (32)$$

where

$$x^- = f_{\lambda+2(n-1)\hbar}^-(u_n) \cdots f_{\lambda+2m\hbar}^-(u_{m+1}), \quad y^- = f_\lambda^-(u_n) \cdots f_{\lambda-2(n-m-1)\hbar}^-(u_{m+1}),$$

$$x^+ = f_{\lambda+2(m-1)\hbar}^+(u_m) \cdots f_\lambda^+(u_1), \quad y^+ = f_{\lambda-2(n-m)\hbar}^+(u_m) \cdots f_{\lambda-2(n-1)\hbar}^+(u_1).$$

The product of zero number of currents is identified with 1 and in this case: $\varepsilon(1) = 1$. The counit ε of a nonzero number of half-currents is always zero. So, this definition generalizes Formulae (30) and (31). We complete the definition of the projections on the subalgebra $\mathcal{A}_F = \mathcal{A}_f \cdot H^+$ by the formulae

$$P_\lambda^+(at^+) = P_\lambda^+(a)t^+, \quad P_\lambda^-(at^+) = P_\lambda^-(a)\varepsilon(t^+),$$

where $a \in \mathcal{A}_f$, $t^+ \in H^+$.

4.3. The projections and the universal elliptic weight function

Consider the expressions of the form

$$P_{\lambda-(n-1)\hbar}^+(f(u_n)f(u_{n-1})\cdots f(u_2)f(u_1)), \quad (33)$$

where the parameter $\lambda - (n-1)\hbar$ is chosen for symmetry reasons. Let us begin with the case $n = 1$. Formula (30) implies that in this case the projection is equal to the positive half-current, which can be represented as an integral transform of the total current:

$$P_\lambda^+(f(u_1)) = f_\lambda^+(u_1) = \oint_{|u_1| > |v_1|} \frac{dv_1}{2\pi i} \frac{\theta(u_1 - v_1 - \lambda)}{\theta(u_1 - v_1)\theta(-\lambda)} f(v_1).$$

\P The integral \oint without limits means a formal integral – a continuous extension of the integral over the unit circle.

The kernel of this transform gives the initial condition for the partition function with a factor:

$$Z^{(1)}(u_1; v_1; \lambda) = \theta(\hbar)\theta(u_1 - v_1) \frac{\theta(u_1 - v_1 - \lambda)}{\theta(u_1 - v_1)\theta(-\lambda)}. \quad (34)$$

The projections (33) can be calculated by generalizing the method proposed in [1] for the algebra $U_q(\hat{\mathfrak{sl}}_2)$. The method uses a recursion over n . Let us first present the last total current in (33) as the difference of half-currents:

$$\begin{aligned} P_{\lambda-(n-1)\hbar}^+(f(u_n) \cdots f(u_2)f(u_1)) &= \\ &= P_{\lambda-(n-3)\hbar}^+(f(u_n) \cdots f(u_2))f_{\lambda-(n-1)\hbar}^+(u_1) - P_{\lambda-(n-1)\hbar}^+(f(u_n) \cdots f(u_2)f_{\lambda-(n-1)\hbar}^-(u_1)). \end{aligned} \quad (35)$$

In the first term we move out the positive half-current from the projection and, therefore, calculation of this term reduces to the computation of the $(n-1)$ -st projection. In the second term in (35) we move the negative half-current to the left step by step using the following commutation relation [2]

$$f(v)f_{\lambda}^-(u_1) = \frac{\theta(v - u_1 - \hbar)}{\theta(v - u_1 + \hbar)}f_{\lambda+2\hbar}^-(u_1)f(v) + \frac{\theta(v - u_1 + \lambda + \hbar)}{\theta(v - u_1 + \hbar)}F_{\lambda}(v),$$

where

$$F_{\lambda}(v) = \frac{\theta(\hbar)}{\theta(\lambda + \hbar)}(f_{\lambda+2\hbar}^+(v)f_{\lambda}^+(v) - f_{\lambda+2\hbar}^-(v)f_{\lambda}^-(v)).$$

At each step we obtain an additional term containing $F_{\lambda}(u)$ and at the last step the negative half-current is annihilated by the projection:

$$P_{\lambda-(n-1)\hbar}^+(f(u_n) \cdots f(u_2)f_{\lambda-(n-1)\hbar}^-(u_1)) = \sum_{j=2}^n Q_j(u_1)X_j, \quad (36)$$

where

$$\begin{aligned} Q_j(u) &= \frac{\theta(u_j - u + \lambda - (n - 2j + 2)\hbar)}{\theta(u_j - u + \hbar)} \prod_{k=2}^{j-1} \frac{\theta(u_k - u - \hbar)}{\theta(u_k - u + \hbar)}, \\ X_j &= P_{\lambda-(n-1)\hbar}^+(f(u_n) \cdots f(u_{j+1})F_{\lambda-(n-2j+3)\hbar}(u_j)f(u_{j-1}) \cdots f(u_2)). \end{aligned}$$

Setting $u_1 = u_i$ in (36), we can substitute the negative half-current for the positive one using the commutation relation for the total currents $f(u)$ and the equality $f(u)f(u) = 0$. Moving out the positive half-current to the left one obtains a linear system of equations for X_i , $i = 2, \dots, n$:

$$P_{\lambda-(n-3)\hbar}^+(f(u_n) \cdots f(u_2))f_{\lambda-(n-1)\hbar}^+(u_i) = \sum_{j=2}^n Q_j(u_i)X_j. \quad (37)$$

Multiplying each equation (37) by

$$\frac{\theta(u_i - u + \lambda)}{\theta(\lambda)} \prod_{k=2}^n \frac{\theta(u_k - u_i + \hbar)}{\theta(u_k - u + \hbar)} \prod_{\substack{k=2 \\ k \neq i}}^n \frac{\theta(u_k - u)}{\theta(u_k - u_i)},$$

summing over $i = 2, \dots, n$ and using the interpolation formula (see Appendix A)

$$Q_j(u) = \sum_{i=2}^n Q_j(u_i) \frac{\theta(u_i - u + \lambda)}{\theta(\lambda)} \prod_{k=2}^n \frac{\theta(u_k - u_i + \hbar)}{\theta(u_k - u + \hbar)} \prod_{\substack{k=2 \\ k \neq i}}^n \frac{\theta(u_k - u)}{\theta(u_k - u_i)} \quad (38)$$

yields

$$\begin{aligned} P_{\lambda-(n-3)\hbar}^+(f(u_n) \cdots f(u_2)) & \sum_{i=2}^n \frac{\theta(u_i - u + \lambda)}{\theta(\lambda)} \prod_{k=2}^n \frac{\theta(u_k - u_i + \hbar)}{\theta(u_k - u + \hbar)} \prod_{\substack{k=2 \\ k \neq i}}^n \frac{\theta(u_k - u)}{\theta(u_k - u_i)} f_{\lambda-(n-1)\hbar}^+(u_i) \\ & = \sum_{j=2}^n Q_j(u) X_j. \end{aligned} \quad (39)$$

Comparing (39) with (36), we conclude that

$$\begin{aligned} P_{\lambda-(n-1)\hbar}^+(f(u_n) \cdots f(u_2) f_{\lambda-(n-1)\hbar}^-(u_1)) & = P_{\lambda-(n-3)\hbar}^+(f(u_n) \cdots f(u_2)) \\ & \times \sum_{i=2}^n \frac{\theta(u_i - u_1 + \lambda)}{\theta(\lambda)} \prod_{k=2}^n \frac{\theta(u_k - u_i + \hbar)}{\theta(u_k - u_1 + \hbar)} \prod_{\substack{k=2 \\ k \neq i}}^n \frac{\theta(u_k - u_1)}{\theta(u_k - u_i)} f_{\lambda-(n-1)\hbar}^+(u_i). \end{aligned} \quad (40)$$

Finally, returning to Formula (35) we derive the following expression for the projection

$$P_{\lambda-(n-1)\hbar}^+(f(u_n) \cdots f(u_2) f(u_1)) = P_{\lambda-(n-3)\hbar}^+(f(u_n) \cdots f(u_2)) f_{\lambda-(n-1)\hbar}^+(u_1; u_n, \dots, u_2), \quad (41)$$

where we introduce the linear combination of the currents:

$$\begin{aligned} f_{\lambda-(n-2m+1)\hbar}^+(u_m; u_n, \dots, u_{m+1}) & = f_{\lambda-(n-2m+1)\hbar}^+(u_m) - \sum_{i=m+1}^n \frac{\theta(u_i - u_m + \lambda + (m-1)\hbar)}{\theta(\lambda + (m-1)\hbar)} \\ & \times \prod_{k=m+1}^n \frac{\theta(u_k - u_i + \hbar)}{\theta(u_k - u_m + \hbar)} \prod_{\substack{k=m+1 \\ k \neq i}}^n \frac{\theta(u_k - u_m)}{\theta(u_k - u_i)} f_{\lambda-(n-2m+1)\hbar}^+(u_i). \end{aligned} \quad (42)$$

Continuing this computation by induction we obtain an expression for the projections in terms of the half-currents (42):

$$P_{\lambda-(n-1)\hbar}^+(f(u_n) \cdots f(u_2) f(u_1)) = \prod_{n \geq m \geq 1}^{\leftarrow} f_{\lambda-(n-2m+1)\hbar}^+(u_m; u_n, \dots, u_{m+1}). \quad (43)$$

Using the addition formula

$$\prod_{i=1}^n G_{\lambda_i}(u_i - v) = \sum_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n G_{\lambda_j}(u_j - u_i) G_{\lambda_0}(u_i - v), \quad (44)$$

where $G_{\lambda}(u - v) = \frac{\theta(u-v+\lambda)}{\theta(u-v)\theta(\lambda)}$, $\lambda_0 = \sum_{i=1}^n \lambda_i$, one can represent the half-currents (42) as integral transforms of the total current:

$$\begin{aligned} f_{\lambda-(n-2m+1)\hbar}^+(u_m; u_n, \dots, u_{m+1}) & = \\ & = \prod_{k=m+1}^n \frac{\theta(u_k - u_m)}{\theta(u_k - u_m + \hbar)} \oint_{|u_i| > |v|} \frac{dv}{2\pi i} \frac{\theta(u_m - v - \lambda - (m-1)\hbar)}{\theta(u_m - v)\theta(-\lambda - (m-1)\hbar)} \prod_{k=m+1}^n \frac{\theta(u_k - v + \hbar)}{\theta(u_k - v)} f(v). \end{aligned}$$

Replacing each combination of the half-currents (42) in (43) by their integral form we obtain

$$P_{\lambda-(n-1)\hbar}^+(f(u_n) \cdots f(u_2)f(u_1)) = \prod_{n \geq k > m \geq 1} \frac{\theta(u_k - u_m)}{\theta(u_k - u_m + \hbar)} \oint_{|u_i| > |v_j|} \frac{dv_n \cdots dv_1}{(2\pi i)^n} \\ \prod_{n \geq k > m \geq 1} \frac{\theta(u_k - v_m + \hbar)}{\theta(u_k - v_m)} \prod_{m=1}^n \frac{\theta(u_m - v_m - \lambda - (m-1)\hbar)}{\theta(u_m - v_m)\theta(-\lambda - (m-1)\hbar)} f(v_n) \cdots f(v_1). \quad (45)$$

Formulae (43) and (45) yield expressions for the universal elliptic weight functions in terms of the current generators of the algebra \mathcal{A} .

4.4. Universal weight function and SOS model partition function

To extract the kernel from the expression (45) and derive a formula for the partition function we use the Hopf pairing (27). Let us calculate the following expression generalizing (34):

$$Z^{(n)}(u_n, \dots, u_1; v_n, \dots, v_1; \lambda) = \prod_{i,j=1}^n \theta(u_i - v_j) \prod_{n \geq k > m \geq 1} \frac{\theta(u_k - u_m + \hbar)\theta(v_k - v_m - \hbar)}{\theta(u_k - u_m)\theta(v_k - v_m)} \\ \times (\hbar\theta(\hbar))^n \left\langle P_{\lambda-(n-1)\hbar}^+(f(u_n) \cdots f(u_1)), e(v_n) \cdots e(v_1) \right\rangle. \quad (46)$$

Using the expression for the projection of the product of the total currents (45) and Formula (28) we obtain

$$Z^{(n)}(u_n, \dots, u_1; v_n, \dots, v_1; \lambda) = \\ = \theta(\hbar)^n \prod_{i,j=1}^n \theta(u_i - v_j) \prod_{k > m} \frac{\theta(v_k - v_m - \hbar)}{\theta(v_k - v_m)} \sum_{\sigma \in S_n} \prod_{\substack{l < l' \\ \sigma(l) > \sigma(l')}} \frac{\theta(v_{\sigma(l)} - v_{\sigma(l')} + \hbar)}{\theta(v_{\sigma(l)} - v_{\sigma(l')} - \hbar)} \times \\ \times \prod_{k > m} \frac{\theta(u_k - v_{\sigma(m)} + \hbar)}{\theta(u_k - v_{\sigma(m)})} \prod_{m=1}^n \frac{\theta(u_m - v_{\sigma(m)} - \lambda - (m-1)\hbar)}{\theta(u_m - v_{\sigma(m)})\theta(-\lambda - (m-1)\hbar)} = \\ = \prod_{k > m} \frac{\theta(v_k - v_m - \hbar)}{\theta(v_k - v_m)} \sum_{\sigma \in S_n} \prod_{\substack{l < l' \\ \sigma(l) > \sigma(l')}} \frac{\theta(v_{\sigma(l)} - v_{\sigma(l')} + \hbar)}{\theta(v_{\sigma(l)} - v_{\sigma(l')} - \hbar)} \times \\ \times \prod_{k > m} \theta(u_k - v_{\sigma(m)} + \hbar) \prod_{k < m} \theta(u_k - v_{\sigma(m)}) \prod_{m=1}^n \frac{\theta(u_m - v_{\sigma(m)} - \lambda - (m-1)\hbar)\theta(\hbar)}{\theta(-\lambda - (m-1)\hbar)}. \quad (47)$$

From this formula we see that the expression (47) defines a holomorphic function of the variables u_i .

Theorem 1 *The set of functions $\{Z^{(n)}(u_n, \dots, u_1; v_n, \dots, v_1; \lambda)\}_{n \geq 1}$ defined by Formula (46) satisfies the conditions of Propositions 1, 2, 3 and the initial condition (22). They coincide with the partition functions of the SOS model with DWBC:*

$$Z_{-+}^{+-}(u_n, \dots, u_1; v_n, \dots, v_1; \lambda) = \\ = \prod_{n \geq k > m \geq 1} \frac{\theta(v_k - v_m - \hbar)}{\theta(v_k - v_m)} \sum_{\sigma \in S_n} \prod_{\substack{l < l' \\ \sigma(l) > \sigma(l')}} \frac{\theta(v_{\sigma(l)} - v_{\sigma(l')} + \hbar)}{\theta(v_{\sigma(l)} - v_{\sigma(l')} - \hbar)} \prod_{1 \leq k < m \leq n} \theta(u_k - v_{\sigma(m)})$$

$$\times \prod_{n \geq k > m \geq 1} \theta(u_k - v_{\sigma(m)} + \hbar) \prod_{m=1}^n \frac{\theta(u_m - v_{\sigma(m)} - \lambda - (m-1)\hbar)\theta(\hbar)}{\theta(-\lambda - (m-1)\hbar)}. \quad (48)$$

The initial condition (22) is satisfied because Formula (34) is satisfied. The first factor in the right-hand side of (46) is symmetric with respect to both sets of variables. The symmetry with respect to the variables $\{u\}$ and the variables $\{v\}$ follows from the commutation relations (26) and (25) respectively. Formula (47) implies that (46) are elliptic polynomials of degree n with character (18) in the variables u_i , in particular in u_n . We now substitute $u_n = v_n - \hbar$ to (47). The non-vanishing terms in the right-hand side correspond to the permutations $\sigma \in S_n$ satisfying $\sigma(n) = n$. Substituting $u_n = v_n - \hbar$ into these terms, one obtains the recursion relation (19). \square

5. Degeneration of the partition function

In this section, we investigate the trigonometric degenerations of the formulae obtained in the elliptic case. In particular, taking the corresponding trigonometric limit in the expression for the SOS model partition function (48) reproduces the expression for the 6-vertex partition function (5).

First we consider the degeneration of the R -matrix, the matrix of Boltzmann weights, which defines the model. To do so we need the formula for the trigonometric degeneration ($\tau \rightarrow i\infty$) of the odd theta function,

$$\lim_{\tau \rightarrow i\infty} \theta(u) = \frac{\sin \pi u}{\pi}.$$

In terms of the multiplicative variables $z = e^{2\pi i u}$, $w = e^{2\pi i v}$, this formula can be rewritten as follows:

$$2\pi i e^{\pi i(u+v)} \lim_{\tau \rightarrow i\infty} \theta(u-v) = z - w.$$

Multiplying the R -matrix (10) by $2\pi i e^{\pi i(u+v)}$ and taking the limit we obtain the following matrix which depends rationally on the multiplicative variables z , w and on the multiplicative parameters $q = e^{\pi i \hbar}$, $\mu = e^{2\pi i \lambda}$:

$$R(z, w; \mu) = 2\pi i e^{\pi i(u+v)} \lim_{\tau \rightarrow i\infty} R(u-v; \lambda) =$$

$$= \begin{pmatrix} zq - wq^{-1} & 0 & 0 & 0 \\ 0 & \frac{(z-w)(\mu q - q^{-1})}{(\mu-1)} & \frac{(z-w\mu)(q - q^{-1})}{(1-\mu)} & 0 \\ 0 & \frac{(z\mu-w)(q - q^{-1})}{(\mu-1)} & \frac{(z-w)(\mu q^{-1} - q)}{(\mu-1)} & 0 \\ 0 & 0 & 0 & (zq - wq^{-1}) \end{pmatrix}. \quad (49)$$

The matrix (49) inherits the property of satisfying the dynamical Yang-Baxter equation and it defines a statistical model called the *trigonometric SOS model*.

To obtain the non-dynamical trigonometric case we need to implement the additional limit $\lambda \rightarrow -i\infty$ implying $\mu \rightarrow \infty$ (or $\lambda \rightarrow i\infty$ implying $\mu \rightarrow 0$):

$$\tilde{R}(z, w) = \lim_{\mu \rightarrow \infty} R(z, w; \mu) = \begin{pmatrix} zq - wq^{-1} & 0 & 0 & 0 \\ 0 & q(z - w) & (q - q^{-1})w & 0 \\ 0 & (q - q^{-1})z & q^{-1}(z - w) & 0 \\ 0 & 0 & 0 & zq - wq^{-1} \end{pmatrix}. \quad (50)$$

The matrix (50) differs from the matrix of the Boltzmann weights of the 6-vertex model (2) by the transformation (21). Taking into account Remark 4 (from Subsection 3.2), we conclude that both matrices (2) and (50) define the same partition function $Z(\{z\}, \{w\})$ with DWBC⁺.

To obtain the partition function with DWBC for the trigonometric SOS model, one should multiply the partition function with DWBC for the elliptic SOS model by a factor and take the trigonometric limit:

$$\begin{aligned} Z_{-+}^{+}(\{z\}, \{w\}; \mu) &= \prod_{k,j=1}^n (2\pi i e^{\pi i(u_k + v_j)}) \lim_{\tau \rightarrow i\infty} Z_{-+}^{+-}(\{u\}, \{v\}; \lambda) = \\ &= \prod_{n \geq k > m \geq 1} \frac{w_k q^{-1} - w_m q}{w_k - w_m} \sum_{\sigma \in S_n} \prod_{\substack{l < l' \\ \sigma(l) > \sigma(l')}} \frac{w_{\sigma(l)} q - w_{\sigma(l')} q^{-1}}{w_{\sigma(l)} q^{-1} - w_{\sigma(l')} q} \\ &\times \prod_{n \geq k > m \geq 1} (z_k q - w_{\sigma(m)} q^{-1}) \prod_{1 \leq k < m \leq n} (z_k - w_{\sigma(m)}) \prod_{m=1}^n \frac{(z_m - w_{\sigma(m)} \mu q^{2(m-1)})(q - q^{-1})}{(1 - \mu q^{2(m-1)})}. \end{aligned} \quad (51)$$

It is easy to prove that Formula (5) is obtained from Formula (51) by taking the limit:

$$Z(\{z\}, \{w\}) = \lim_{\mu \rightarrow \infty} Z_{-+}^{+-}(\{z\}, \{w\}; \mu).$$

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⁺ The matrix (50) is the limit of a matrix which differs from (10) by the transformation (21) with $\rho = q$.

Appendix A. Interpolation formula for elliptic polynomials

A group homomorphism $\chi: \Gamma \rightarrow \mathbb{C}^\times$, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ and \mathbb{C}^\times is the multiplicative group of nonzero complex numbers is called a *character*. Each character χ and integer number n define a space $\Theta_n(\chi)$ consisting of the holomorphic functions on \mathbb{C} with the translation properties

$$\phi(u+1) = \chi(1)\phi(u), \quad \phi(u+\tau) = \chi(\tau)e^{-2\pi i n u - \pi i n \tau} \phi(u).$$

If $n > 0$ then $\dim \Theta_n(\chi) = n$ (and $\dim \Theta_n(\chi) = 0$ if $n < 0$). The elements of the space $\Theta_n(\chi)$ are called elliptic polynomials (or theta-functions) of degree n with character χ .

Proposition 4 *Let $\{\phi_j\}_{j=1}^n$ be a basis of $\Theta_n(\chi)$, with character $\chi(1) = (-1)^n$, $\chi(\tau) = (-1)^n e^{2\pi i \alpha}$, then the determinant of the matrix $\|\phi_j(u_i)\|_{\leq i, j \leq n}$ is equal to*

$$\det \|\phi_j(u_i)\| = C \cdot \theta\left(\sum_{k=1}^n u_k - \alpha\right) \prod_{i < j} \theta(u_i - u_j), \quad (\text{A.1})$$

where C is a nonzero constant.

Consider the ratio

$$\frac{\det \|\phi_j(u_i)\|}{\theta\left(\sum_{k=1}^n u_k - \alpha\right) \prod_{i < j} \theta(u_i - u_j)}. \quad (\text{A.2})$$

This is an elliptic function of each u_i with only simple poles in any fundamental domain (the points u_i satisfying $\sum_{k=1}^n u_k - \alpha \in \Gamma$). Therefore, it is a constant function of each u_i . Thus this ratio does not depend on u_i and we have to prove that it does not vanish, that is that the determinant $\det \|\phi_j(u_i)\|$ is not identically zero. Let us denote by $\Delta_{j_1, \dots, j_k}^{i_1, \dots, i_k}$ the minor of this determinant corresponding to the i_1 -th, \dots , i_k -th rows and the j_1 -th, \dots , j_k -th columns. Suppose that this determinant is identically zero and consider the following decomposition

$$\det \|\phi_j(u_i)\| = \sum_{k=1}^n (-1)^{k+1} \phi_k(y_1) \Delta_{1, \dots, k-1, k+1, \dots, n}^{2, \dots, n}. \quad (\text{A.3})$$

Since the functions $\phi_k(y_1)$ are linearly independent, the minors $\Delta_{1, \dots, k-1, k+1, \dots, n}^{2, \dots, n}$ are identically zero. Decomposing the minor $\Delta_{2, \dots, n}^{2, \dots, n}$ we conclude that the minors $\Delta_{2, \dots, k-1, k+1, \dots, n}^{3, \dots, n}$ are identically zero, and so on. Finally, we obtain that $\Delta_n^n = \phi_n(y_n)$ is identically zero which cannot be true. \square

Lemma 2 *Let us consider two elliptic polynomials $P_1, P_2 \in \Theta_n(\chi)$, where $\chi(1) = (-1)^n$, $\chi(\tau) = (-1)^n e^\alpha$, and n points u_i , $i = 1, \dots, n$, such that $u_i - u_j \notin \Gamma$, $i \neq j$, and $\sum_{k=1}^n u_k - \alpha \notin \Gamma$. If the values of these polynomials coincide at these points, $P_1(u_i) = P_2(u_i)$, then these polynomials coincide: $P_1(u) = P_2(u)$.*

Decomposing the polynomials under consideration as $P_a(u) = \sum_{i=1}^n p_a^i \phi_i(u)$, $a = 1, 2$, we obtain the system of equations

$$\sum_{i=1}^n p_{12}^i \phi_i(u) = 0,$$

with respect to the variables $p_{12}^i = p_1^i - p_2^i$. We have proved that the determinant of this system is equal to (A.1) and therefore is not zero. Hence, this system has only the trivial solution $p_{12}^i = 0$, but this implies $P_1(u) = P_2(u)$. \square

Let $P \in \Theta_n(\chi)$ be an elliptic polynomial, where $\chi(1) = (-1)^n$, $\chi(\tau) = (-1)^n e^{2\pi i \alpha}$, and u_i , $i = 1, \dots, n$, be n points such that $u_i - u_j \notin \Gamma$, $i \neq j$, and $\sum_{k=1}^n u_k - \alpha \notin \Gamma$. This polynomial can be recovered from the values at these points:

$$P(u) = \sum_{i=1}^n P(u_i) \frac{\theta(u_i - u + \alpha - \sum_{m=1}^n u_m)}{\theta(\alpha - \sum_{m=1}^n u_m)} \prod_{\substack{k=1 \\ k \neq i}}^n \frac{\theta(u_k - u)}{\theta(u_k - u_i)}. \quad (\text{A.4})$$

Indeed, the right hand side belongs to $\Theta_n(\chi)$, this equality holds at the points $u = u_i$. Using Lemma 2, we conclude that (A.4) holds at all $u \in \mathbb{C}$.

Consider the meromorphic functions

$$Q_j(u) = \frac{\theta(u_j - u + \lambda - (n - 2j + 2)\hbar)}{\theta(u_j - u + \hbar)} \prod_{k=2}^{j-1} \frac{\theta(u_k - u - \hbar)}{\theta(u_k - u + \hbar)}.$$

It is easy to check that the functions

$$\begin{aligned} P_j(u) &= \prod_{k=2}^n \theta(u_k - u + \hbar) Q_j(u) = \\ &= \theta(u_j - u + \lambda - (n - 2j + 2)\hbar) \prod_{k=2}^{j-1} \theta(u_k - u - \hbar) \prod_{k=j+1}^n \theta(u_k - u + \hbar) \end{aligned}$$

belong to $\Theta_{n-1}(\chi)$, where $\chi(1) = (-1)^{n-1}$, $\chi(\tau) = (-1)^{n-1} e^{2\pi i \alpha}$, $\alpha = \lambda + \sum_{k=2}^n u_k$. Since $\lambda \notin \Gamma$, the polynomials $P_j(u)$ can be recovered from by their values $P_j(u_i)$ via the interpolation formula (A.4). Taking into account the relation between $Q_j(u)$ and $P_j(u)$ we obtain Formula (38). *

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* We can require the condition $u_i - u_j \notin \Gamma$, because the u_i 's in Formula (38) are formal variables.

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